233. (**June, 1915.**) **Proposed by V. M. SPUNAR, Chicago, Illinois.**

Solve in rational numbers $x^2 + y^2 = a^2$, $xy = m/n$, when m and n are integers and relatively prime to each other.

SOLUTION BY A. H. HOLMES, Brunswick, Maine.

Putting $x = (p^2 - q^2)/(2p \pm 2q + 1)$, $y = (2pq)/(2p \pm 2q + 1)$, we shall have

$$a = (p^2 + q^2)/(2p \pm 2q + 1), \qquad m = 2pq(p^2 - q^2),$$

and $n = (2p \pm 2q + 1)^2$ except in the cases where $p = q + 1, 2q + 1$, or $5q + 1$, in which cases $n = 2p \pm 2q + 1$.

For smallest values put $q = 1$, $p = 2$, using the minus sign of the denominator. Then $x = 1$, $y = 4/3$, $a = 5/3$, $m = 4$, and $n = 3$.

Also solved by H. N. CARLETON.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

I. RELATING TO AN INSTRUCTIVE PROBLEM IN ATTRACTION.

BY EDWIN BIDWELL WILSON, Massachusetts Institute of Technology.

In B. O. Pierce's *Newtonian Potential Function* (3d ed.) there is found on page 28 this exercise:

Show that the attraction at the focus of a segment of a paraboloid of revolution bounded by a plane perpendicular to the axis at a distance b from the vertex is of the form $4\pi\rho a \log(1 + b/a)$.

This problem is also found in other texts. The answer is very easy to get; it is also wrong. One way in which the answer may be found is this:

The attraction of a disc of mass M and radius r at a point on its axis at a distance c from the center is

$$\frac{2M}{r^2}\left(1 - \frac{c}{\sqrt{c^2 + r^2}}\right).$$

If the equation of the revolved parabola is $4ay = x^2$, the attraction of a segment at the focus is

$$\frac{2\pi\rho \cdot 4ay}{4ay}\left(1 - \frac{y - a}{\sqrt{(y - a)^2 + 4ay}}\right)dy,$$

and the total attraction is

$$A = 2\pi\rho \int_0^b \left(1 - \frac{y - a}{\sqrt{(y + a)^2}}\right)dy.$$

To the careless person this gives

$$A = 2\pi\rho \int_0^b \frac{2ady}{y + a} = 4\pi\rho a \log(1 + b/a)$$

because $\sqrt{(y+a)^2}$ for him equals $y+a$. Indeed, since $y+a$ does not pass through zero, it might seem as though the root could be extracted in this way. In the numerator, however, there occurs the expression $y-a$ which does pass through zero when $y=a$ (*i. e.*, at the section through the focus) and this should be a danger signal.

There is another sign of trouble and a very fundamental one. We can not expect to get a single formula to express the attraction of the segment for the two cases $b < a$ and $b > a$, any more than we can expect to get a single formula for the attraction of a sphere on a point irrespective of whether the point is inside or outside of the sphere. The attraction is a continuous function of b; but the derivative of the force with respect to b is discontinuous when $b=a$, as is familiar to all who are acquainted with Poisson's Equation and its physical significance. The amount of the discontinuity is $4\pi\rho$.

The solution of this, and similar problems, is carried out in two steps. First when $b < a$, the attraction is toward the vertex of the paraboloid and equal to

$$A = 2\pi\rho\int_0^b\left(1-\frac{a-y}{a+y}\right)dy = 4\pi\rho a\left[\frac{b}{a}-\log\left(\frac{b}{a}+1\right)\right],$$

which, when $b=a$, gives the value $4\pi\rho a(1-\log 2)$. When $b > a$, the calculation of the attraction of the part of the segment beyond the focus gives

$$2\pi\rho\int_a^b\left(1-\frac{y-a}{y+a}\right)dy = 4\pi\rho a\left[\log\left(\frac{b}{a}+1\right)-\log 2\right]$$

and directed away from the vertex. The whole segment from 0 to b, therefore, has the attraction equal to

$$A = 4\pi\rho a\left[1-\log\left(\frac{b}{a}+1\right)\right], \qquad b > a,$$

toward the vertex. This value vanishes when $b = a(e-1)$ or when $b-a = (e-2)a = 0.718a$. When b exceeds $a(e-1)$ the resultant attraction is away from the vertex.

II. Relating to a Problem in Minima.

By Dunham Jackson, Harvard University.

On page 339 of Osgood's Calculus (Ex. 9), the following problem is proposed: "Find the point so situated that the sum of its distances from the three vertices of an acute-angled triangle is a minimum."

The answer is given:

"The lines joining the point with the vertices make angles of 120° with one another."

Then the note is added,

"For a complete discussion of the problem for any triangle see Goursat-Hedrick, *Mathematical Analysis*, Vol. 1, § 62."